Analysis of the self-similar spreading of power law fluids

D. G. Aronson,[*] S. I. Betelú,[†] M. A. Fontelos, A. Sánchez[‡]

February 7, 2008

Abstract

We consider the equation that models the spreading of thin liquid films of power-law rheology. In particular, we analyze the existence and uniqueness of source-type self-similar solutions in planar and circular symmetries. We find that for shear-thinning fluids there exist a family of such solutions representing both finite and zero contact angle drops and that the solutions with zero contact angle are unique. We also prove the existence of traveling waves in one space dimension and classify them.

1 Introduction

Here we study capillary spreadings of thin films of liquids of power-law rheology, also known as Ostwald-de Waele fluids [8]. The power-law rheology is one of the simplest generalizations of the Newtonian one, in which the effective viscosity η at a point is assumed to be a function of the local rate of deformation $\dot{\gamma}$ given by $\eta = m|\dot{\gamma}|^{1/\lambda-1}$. The values of m and λ depend on the physical properties of the liquid. When $\lambda > 1$ the viscosity tends to zero at high strain rates [8] and is larger at low strain rates (these fluids are called shear-thinning).

In [10, 6] the following equation for one dimensional motion was derived using the lubrication approximation:

$$u_t + (u^{\lambda+2}|u_{xxx}|^{\lambda-1}u_{xxx})_x = 0. \tag{1}$$

where $u(x,t)$ represents the thickness of the one-dimensional liquid film at position x and time t. In [1] a generalized version of (1) was studied by means of asymptotic and perturbative techniques in order to construct approximate solutions representing the spreading of a droplet. We will look for solutions

[*]Mathematics Department, University of Minnesota, 127 Vincent Hall, 206 Church street, Minneapolis, MN-55444, USA.

[†]Mathematics Department, University of North Texas, P.O. Box 311430, Denton, TX 76203-1430, USA.

[‡]Departamento de Matemática Aplicada, Universidad Rey Juan Carlos, C/ Tulipán S/N, Móstoles 28933, Madrid, Spain.

with a compact support $[x_1(t), x_2(t)]$. Therefore, by conservation of mass and equation (1),

$$0 = \frac{d}{dt} \int_{x_1(t)}^{x_2(t)} u(x,t)dx$$

$$= x_2'(t)u(x_2(t),t) - x_1'(t)u(x_1(t),t) + \left(u^{\lambda+2}|u_{xxx}|^{\lambda-1}u_{xxx}\right)\Big|_{x_1(t)}^{x_2(t)}$$

which forces us to impose

$$x_i'(t)u(x_i(t),t) + \left(u^{\lambda+2}|u_{xxx}|^{\lambda-1}u_{xxx}\right)(x_i(t),t) = 0 \;,\; i = 1,2. \tag{2}$$

Equation (1) formally admits solutions of the form

$$u(x,t) = \frac{A}{t^\beta}U\left(\frac{x}{t^\beta}\right) \tag{3}$$

with

$$\beta = \frac{1}{5\lambda+2}. \tag{4}$$

Without loss of generality, one can set $A^{2\lambda+1} = 1/(5\lambda+2)$. Then with $\eta = x/t^\beta$, $U(\eta)$ satisfies the ODE,

$$U^{\lambda+2}U'''^{\lambda} = \eta U \;, \tag{5}$$

which results from introducing (3) into (1), integrating once and choosing the integration constant $K = 0$ (by(2)).

The initial conditions for symmetric drops are

$$U(0) = 1, \qquad U'(0) = 0, \qquad U''(0) = -\kappa, \tag{6}$$

where κ is a real positive parameter.

For radially symmetric flows, the equation of motion is (cf. [7]):

$$u_t + \frac{1}{r}\left(ru^{\lambda+2}\left|\left(u_{rr} + \frac{u_r}{r}\right)_r\right|^{\lambda-1}\left(u_{rr} + \frac{u_r}{r}\right)_r\right)_r = 0,$$

where r is the radial coordinate, and the self-similar solutions are of the form

$$u(r,t) = \frac{A}{t^{2\beta}}U\left(\frac{r}{t^\beta}\right)$$

with

$$\beta = \frac{1}{7\lambda+3}$$

and $U(\eta)$ satisfying the following ordinary differential equation:

$$U^{\lambda+2}\left(\left(U'' + \frac{U'}{\eta}\right)'\right)^\lambda = \eta U \;. \tag{7}$$

Analogously to (6) we impose

$$U(0) = 1, \qquad U'(0) = 0, \qquad \lim_{\eta \to 0^+} \frac{1}{\eta}(\eta U')' = -\kappa \ . \qquad (8)$$

For simplicity and clarity we begin with a datached analysis of the existence and uniqueness of one-dimensional self-similar solutions, i.e., solutions to Eqs. (5-6). This is done in section 2. In section 3 we indicate the changes needed to deal with radially symmetric self-similar solutions (Eqs. (7-8)). Finally, in an Appendix we study the traveling wave solutions of (1) and classify all the possible behaviors of moving fronts close to the interface.

2 Analysis of one dimensional self-similar solutions

In order to remove the parameter κ from the initial condition we introduce the change of variables

$$x = \eta\sqrt{\kappa}. \qquad (9)$$

Then $z(x, \gamma) = U(\eta)$ satisfies

$$z^{1+a}z''' = \gamma x^a \qquad (10)$$

$$z(0) = 1, z'(0) = 0, z''(0) = -1,$$

with

$$\gamma = k^{-(3+1/\lambda)/2} \qquad (11)$$

and

$$a = 1/\lambda < 1.$$

In the following subsections we prove:

Theorem 1 For each $a \in (0, 1)$ there exists a $\gamma = \gamma(a)$ and $y = y(a)$ such that $z(x, \gamma(a))$ first reaches $z = 0$ for $x = y(a)$ and $z'(y(a), \gamma(a)) = 0$. The function $z(x, \gamma(a))$ satisfying these conditions is unique.

Theorem 2 Given $a \in (0, 1)$ and $\theta \in (0, 1)$ there exist $\gamma = \gamma_\theta$ and $y = y_\theta$ such that $z(x, \gamma_\theta)$ first reaches $z = 0$ for $x = y_\theta$ and $z'(y_\theta, \gamma_\theta) = -\sqrt{2}\theta$.

These results are physically meaningful, because they imply that for $\lambda > 1$ there exist solutions describing one-dimensional drops with fronts advancing at finite speed. This result contrasts with the Newtonian case $\lambda = 1$, where such solutions do not exist [3].

We also show that source-type self-similar solutions cannot exist for $\lambda \leq 1$.

All these results were suggested by numerical calculations in [6] and asymptotic analysis in [6, 10].

In an appendix we will show, by studying traveling wave solutions, that the local behaviors of moving fronts near the interface are just those described in Theorems 1 and 2.

3

2.1   General Properties of Solutions.

We consider the initial value problem

$$z^{1+a}z''' = \gamma x^a \tag{12}$$

$$z(0) = 1, z'(0) = 0, z''(0) = -1,$$

where $a \geq 0$ is fixed and $\gamma \in \mathbf{R}$ is a parameter. We are interested in finding those values of γ for which there are interfaces, i.e., such that the solution $z = z(x, \gamma)$ has a zero for a finite value of x and γ.

If $\gamma = 0$ then the non-trivial solution to (12) is

$$z(x, 0) = 1 - \frac{x^2}{2} \tag{13}$$

which decreases from 1 to 0 as x increases from 0 to $\sqrt{2}$. The interface is $x = \sqrt{2}$.

If $\gamma < 0$ then the first three x-derivatives of $z(x, \gamma)$ are negative whenever $z > 0$. Thus z is a decreasing function of x as long as z is positive. Moreover, $z''' < 0$ and the initial conditions imply that $z(x, \gamma) < z(x, 0)$ for $x > 0$. Physically, the case $\gamma < 0$ represents retracting, dewetting drops.

If $\gamma > 0$ then $z(x, \gamma) > z(x, 0)$ for $x > 0$. Moreover, $z'(x, \gamma) < 0$ at least for all sufficiently small $x > 0$. Either $z'(x, \gamma) < 0$ for all x such that $z(x, \gamma) > 0$ or else there exists a $\zeta > 0$ such that $z(x, \gamma)$ decreases on $(0, \zeta)$ and achieves a minimum at $x = \zeta$ with $z(x,\gamma) \in (0, 1)$. At $x = \zeta$ we have $z'(\zeta, \gamma) = 0$ and $z''(\zeta, \gamma) \geq 0$. Since $z''' > 0$ it follows that $z''(x, \gamma) > 0$ and $z'(x, \gamma) > 0$ for all $x > \zeta$. Therefore if z is not everywhere decreasing for $z > 0$ then z has a unique minimum at some $x = \zeta > 0$ with

$$0 < z(\zeta, \gamma) < z(x, \gamma) < 1, \ \ z'(x, \gamma) < 0 \text{ for } x \in (0, \zeta)$$

and

$$z(\zeta, \gamma) < z(x, \gamma), \ \ z'(x, \gamma) > 0 \text{ for } \ x \in (\zeta, \infty).$$

2.2   Local Expansions Near the Interface

There are two different asymptotic expansions for the solution near the root $z(y, \gamma) = 0$: one represents solutions with finite contact angle,

$$z(x, \gamma) = \sqrt{2}\theta(y - x) + \frac{\gamma y^a}{2^{\frac{1+a}{2}}\theta^{1+a}a(1-a)(2-a)}(y - x)^{2-a} + \text{lower order terms} \tag{14}$$

where $\theta \in (0, 1)$, and another with zero contact angle

$$z(x, \gamma) = \left(\frac{\gamma(2+a)^3}{3a(1-a)(1+2a)}\right)^{1/(2+a)} y^{a/(2+a)}(y-x)^{3/(2+a)} + \text{lower order terms}. \tag{15}$$

4

It is simple to show the nonexistence of local expansions near the interface if $a \geq 1$ and $\gamma > 0$. If $a \geq 1$ and $\gamma \leq 0$, then the local expansion is given by (14).

We can write formal representation formulae for the solution to (12). Let

$$I_1(x, \gamma) = \int_0^x \frac{\xi^a d\xi}{z^{1+a}(\xi, \gamma)}$$

and

$$I_{j+1}(x, \gamma) = \int_0^x I_j(\zeta, \gamma) d\zeta$$

for $j = 1, 2$. Then as long as $z > 0$

$$z''(x, \gamma) = -1 + \gamma I_1(x, \gamma), \tag{16}$$

$$z'(x, \gamma) = -x + \gamma I_2(x, \gamma), \tag{17}$$

and

$$z(x, \gamma) = 1 - \frac{x^2}{2} + \gamma I_3(x, \gamma). \tag{18}$$

It is clear from (14) and (15) that if $a < 1$ then the representation formulae (17) and (18) can be extended to the interface. In particular, if $z(x, \gamma) > 0$ for $0 \leq x < y$ and $z(x, \gamma) \to 0$ as $x \nearrow y$ then

$$z''(x, \gamma) \to +\infty, \tag{19}$$

$$z'(x, \gamma) \to z'(y, \gamma) = -y + \gamma I_2(y, \gamma), \tag{20}$$

and

$$z(x, \gamma) \to z(y, \gamma) = 1 - \frac{y^2}{2} + \gamma I_3(y, \gamma). \tag{21}$$

If $a \geq 1$ then (19) and (21) are still valid, but (20) is replaced by $z'(x, \gamma) \to +\infty$. Hence, there are no interfaces for $a \geq 1$ unless $\gamma \leq 0$.

There are no interfaces for sufficiently large values of γ. More precisely, we prove

Lemma 1. *There are no solutions with an interface if*

$$\gamma > \left\{ \frac{1+a}{2(3+a)} \right\}^{\frac{1+a}{2}} (1+a)(2+a). \tag{22}$$

Proof. As we observed above, any positive local minimum of $z(x, \gamma)$ is in fact a unique global minimum. Thus if z has a positive minimum, then z does not have an interface. Suppose z has a positive minimum at $x = x_0$. Then since $0 < z < 1$, we have

$$z(x, \gamma) = 1 - \frac{x^2}{2} + \gamma I_3(x, \gamma) \geq 1 - \frac{x^2}{2} + \frac{\gamma x^{3+a}}{(1+a)(2+a)(3+a)} \equiv P(x, \gamma)$$

for $x \in [0, x_0]$. The condition (22) guarantees that the positivity of the minimum value of P. Therefore if (22) is satisfied, z is everywhere positive and has no interface.□

2.3   Analysis for Small γ

Since (12) is singular at $z = 0$, we shall first study the behavior of the solutions for $z > 0$, and, instead of studying the interfaces $z(y, \gamma) = 0$ directly, we first solve

$$z(y, \gamma) = \delta \tag{23}$$

for given $\delta \in (0, 1)$. In the next subsection we study the limit as $\delta \searrow 0$. We begin by studying (a) in the neighborhood of the solution (13) for small $|\gamma|$.

We have

$$z(y_1(\delta), 0) = \delta \text{ and } z'(y_1(\delta), 0) = -y_1(\delta) < 0,$$

where $y_1(\delta) = \sqrt{2(1 - \delta)}$. Since $z(x, \gamma)$ is smooth for $z > 0$, by the Implicit Function Theorem there exists a smooth function $y(\gamma, \delta)$ for sufficiently small $|\gamma|$ such that

$$y(0, \delta) = y_1(\delta) \text{ and } z(y(\gamma, \delta), \gamma) = \delta.$$

$y(\gamma, \delta)$ exists for all γ such that

$$z'(y(\gamma, \delta), \gamma) < 0. \tag{24}$$

As we observed above, (24) holds for all $\gamma \leq 0$. Moreover, since (24) holds for $\gamma = 0$, and since $y(\gamma, \delta)$ depends smoothly on γ and $z(y, \gamma)$ depends smoothly on y it follows that (24) continues to hold for all sufficiently small $\gamma > 0$. Differentiating (23) with respect to γ we find

$$z'(y, \gamma) \frac{\partial y}{\partial \gamma} + z_\gamma(y, \gamma) = 0.$$

Therefore

$$\frac{\partial y}{\partial \gamma} \Big|_{\gamma=0} = -\frac{z_\gamma(y_1(\delta), 0)}{z'(y_1(\delta), 0)} = \frac{I_2(y_1(\delta), 0)}{\sqrt{2(1 - \delta)}} > 0$$

so that y is an increasing function of γ at least for $|\gamma|$ small.

2.4   Estimates for solutions with interfaces

We now investigate the maximal γ-interval of existence for the functions $y(\gamma, \delta)$ constructed above. In the limit as $\delta \to 0$ this will lead to solutions with zero contact angle.

In view of (24) we define

$$\gamma_0(\delta) \equiv \sup \{\gamma > 0 : z'(y(\gamma, \delta), \gamma) < 0\}$$

and

$$y_0(\delta) \equiv \lim_{\gamma \nearrow \gamma_0(\delta)} y(\gamma, \delta).$$

6

Note that since $z'(y, \gamma)$ and $y(\gamma, \delta)$ are continuous we have

$$z(y_0(\delta), \gamma_0(\delta)) = \delta \text{ and } z'(y_0(\delta), \gamma_0(\delta)) = 0.$$

We now derive some estimates for $\gamma_0(\delta)$ and $y_0(\delta)$.

Lemma 2. *Let*

$$B(\gamma) = \left\{ \frac{(1+a)(2+a)}{\gamma} \right\}^{\frac{1}{1+a}} \text{ and } G = 2^{\frac{1+a}{2}}(1+a)(2+a).$$

Then

$$0 < \gamma_0(\delta) < G \tag{25}$$

and

$$\sqrt{2(1-\delta)} < y_0(\delta) < B(\gamma_0(\delta)). \tag{26}$$

Proof. For $0 < x < y(\gamma, \delta)$ we have $\delta < z(x, \gamma) < 1$, and $z'(y(\gamma, \delta), \delta) < 0$. It follows from (20) that

$$0 > z'(y(\gamma, \delta), \gamma) = -y(\gamma, \delta) + \gamma I_2(y(\gamma, \delta), \gamma) > -y(\gamma, \delta) + \frac{\gamma y^{1+a}(\gamma, \delta)}{(1+a)(2+a)}.$$

Therefore

$$\sqrt{2(1-\delta)} < y(\gamma, \delta) < B(\gamma).$$

Letting $\gamma \nearrow \gamma_0(\delta)$ we immediately obtain (26). Then (25) follows from $\sqrt{2(1-\delta)} < B(\gamma_0(\delta))$. $\square$

For arbitrary fixed $\theta \in (0, 1)$ consider

$$\Delta(\gamma, \theta) \equiv z'(y(\gamma, \delta), \gamma) + \theta\sqrt{2(1-\delta)}.$$

$\Delta(\gamma, \theta)$ is a continuous function of $\gamma \in [0, \gamma_0(\delta)]$ with

$$\Delta(0, \theta) = -(1-\theta)\sqrt{2(1-\delta)} < 0 \text{ and } \Delta(\gamma_0(\delta), \theta) = \theta\sqrt{2(1-\delta)} > 0.$$

By the Intermediate Value Property of continuous functions there exists $\gamma(\delta, \theta) \in (0, \gamma_0(\delta))$ such that $\Delta(\gamma(\delta, \theta), \theta) = 0$, i.e.,

$$z'(Y(\delta, \theta), \gamma(\delta, \theta)) = -\theta\sqrt{2(1-\delta)}, \tag{27}$$

where

$$Y(\delta, \theta) = y(\gamma(\delta, \theta), \delta).$$

We take $\gamma(\delta, \theta)$ to be the smallest positive value of γ for which (27) holds. We extend the definition of $\gamma(\delta, \theta)$ by setting $\gamma(\delta, 0) = \gamma_0(\delta)$ and $\gamma(\delta, 1) = 0$. In view of (25)

$$\gamma_\theta \equiv \liminf_{\delta \searrow 0} \gamma(\delta, \theta)$$

exists and satisfies

$$0 \leq \gamma_\theta \leq G$$

7

for arbitrary $\theta \in [0,1]$. For each $\theta \in [0,1]$ fix a sequence $\{\delta_\theta^j\}$ such that $\delta_\theta^j \searrow 0$ and $\gamma_\theta^j \equiv \gamma(\delta_\theta^j, \theta) \to \gamma_\theta$ as $j \to \infty$.

Lemma 3. $\gamma_\theta > 0$ if and only if $a < 1$.

Proof. From (17) we have

$$-\theta \sqrt{2(1-\delta)} = z'(Y(\delta,\theta), \gamma(\delta,\theta)) = -Y(\delta,\theta) + \gamma(\delta,\theta) I(\delta,\theta)$$

or

$$Y(\delta,\theta) - \theta\sqrt{2(1-\delta)} = \gamma(\delta,\theta) I(\delta,\theta),$$

where

$$I(\delta,\theta) = I_2(Y(\delta,\theta), \gamma(\delta,\theta)).$$

Suppose that $\gamma_\theta = 0$. Then since

$$(1-\theta)\sqrt{2(1-\delta_\theta^j)} < \gamma_\theta^j I(\delta_\theta^j, \theta),$$

$\gamma(\delta_\theta^j, \theta) \to 0$ implies $I(\delta_\theta^j, \theta) \to \infty$. In view of the asymptotic behavior of z near 0 this can only occur if $a \geq 1$. On the other hand, if $\gamma(\delta_\theta^j, \theta) > 0$ then

$$\gamma(\delta,\theta) I(\delta,\theta) < Y(\delta,\theta) < B(\gamma(\delta,\theta))$$

so that

$$I(\delta_\theta^j, \theta) < \frac{B(\gamma_\theta^j)}{\gamma_\theta^j}.$$

Therefore

$$0 < \lim_{j\to\infty} \sup I(\delta_\theta^j, \theta) \leq \frac{B(\gamma_\theta)}{\gamma_\theta} < \infty$$

which can occur only if $a < 1$.□

We now investigate the behavior of $Y(\delta_\theta^j, \theta)$ as $j \to \infty$.

Lemma 4. *If* $a < 1$ *then*

$$\sqrt{2} \leq y_\theta \equiv \lim_{j\to\infty} Y(\delta_\theta^j, \theta) \leq B(\gamma_\theta).$$

Proof. $\gamma_\theta^j \to \gamma_\theta$ implies that for arbitrary $\varepsilon \in (0, \gamma_\theta)$ there exists an $N = N(\varepsilon) > 0$ such that $\gamma_\theta^j > \gamma_\theta - \varepsilon$ for all $j > N$. In view of (26)

$$\sqrt{2(1-\delta_\theta^j)} < Y(\delta_\theta^j, \theta) \leq B(\gamma_\theta - \varepsilon).$$

Thus there is a subsequence of the δ_θ^j which we again call $\{\delta_\theta^j\}$ along which $Y(\delta_\theta^j, \theta)$ converges to a limit y_θ which satisfies

$$\sqrt{2} \leq y_\theta \leq B(\gamma_\theta - \varepsilon).$$

Since ε is arbitrary the assertion now follows by letting $\varepsilon \searrow 0$.□

Finally we prove

8

Lemma 5. *If* $a < 1$ *then*

$$z(y_\theta, \gamma_\theta) = \lim_{j \to \infty} z(Y(\delta_\theta^j, \theta), \gamma_\theta) = 0 \text{ and } z'(y_\theta, \gamma_\theta) = \lim_{j \to \infty} z'(Y(\delta_\theta^j, \theta), \gamma_\theta) = -\theta\sqrt{2}.$$

Proof. From the representation formulas (17) we have

$$z(Y(\delta_\theta^j, \theta), \gamma_\theta) = 1 - \frac{Y^2(\delta_\theta^j, \theta)}{2} + \gamma_0 I_3(Y(\delta_\theta^j, \theta), \gamma_\theta)$$

which we rewrite in the form

$$z(Y(\delta_\theta^j, \theta), \gamma_\theta) = \delta_\theta^j + \gamma_\theta \left\{ I_3(Y(\delta_\theta^j, \theta), \gamma_\theta) - I_3(Y(\delta_\theta^j, \theta), \gamma_\theta^j) \right\}$$
$$+ \left\{ \gamma_\theta - \gamma_\theta^j \right\} I_3(Y(\delta_\theta^j, \theta), \gamma_\theta^j).$$

The first and third terms on the right hand side clearly tend to zero as $j \to \infty$. The convergence of the middle term to zero follows from the asymptotic behavior of z near zero and the Lebesgue Dominated Convergence Theorem. The result for z' is proved in a similar manner.□

Corollary. *If* $a < 1$ *then* $y_\theta > \sqrt{2}$.

Proof. Since $\gamma_\theta > 0$ we have $z'''(x, \gamma_\theta) > 0$ on $(0, y_\theta)$. Therefore

$$0 = z(y_\theta, \gamma_\theta) > 1 - \frac{y_\theta^2}{2}$$

which implies that $y_\theta > \sqrt{2}$.□

The analytic considerations given above establish the existence for $a < 1$ of solutions whose interfaces have zero contact angle and of solutions whose interfaces have non-zero contact angles. In particular, for each $\theta \in [0,1]$ there is a pair $(y_\theta, \gamma_\theta)$ such that the solution $z(x, \gamma_\theta)$ has its interface at $x = y_\theta$ with $z'(y_\theta, \gamma_\theta) = -\theta\sqrt{2}$. However, the results so far do not give any global information about uniqueness and monotonicity of $(y_\theta, \gamma_\theta)$. In the next subsection we will prove the uniqueness of zero contact angle solutions for each $\gamma > 0$.

2.5 Uniqueness of zero contact angle solutions

In this section we present a proof of uniqueness of the zero contact angle solution following arguments similar to those used in [4] in a somewhat different context. Rewrite equation (5) in the form

$$U''' = \eta^{\frac{1}{\lambda}} U^{-s}, \tag{28}$$

where $s = 1 + 1/\lambda$, and let $U_j(\eta)$ for $j = 1, 2$ be solutions to (28) such that

$$U_j(0) = 1, U_j'(0) = 0,$$

and
$$U_j(\eta_j) = U_j'(\eta_j) = 0.$$

Without loss of generality we assume that
$$\eta_1 \geq \eta_2.$$

Set
$$c = \left(\frac{\eta_1}{\eta_2}\right)^{\frac{3\lambda+1}{2\lambda+1}} \quad \text{and} \quad \xi = \frac{\eta\eta_1}{\eta_2}.$$

Note that $c \geq 1$. The rescaled function
$$\widetilde{U}_2(\xi) = cU_2(\eta)$$

is a solution to (11) with
$$\widetilde{U}_2(0) = c, \widetilde{U}_2'(0) = 0,$$

and
$$\widetilde{U}_2(\eta_1) = \widetilde{U}_2'(\eta_1^-) = 0.$$

For $\xi \in [0, \eta_1]$ define
$$V(\xi) \equiv \widetilde{U}_2(\xi) - U_1(\xi).$$

Then
$$V(0) = c - 1 \geq 0, V'(0) = 0, \text{ and } V(\eta_1) = 0.$$

Moreover, in view of (28),
$$V''' = \xi^{\frac{1}{\lambda}} \left\{ f(\widetilde{U}_2) - f(U_1) \right\},$$

where
$$f(\tau) = \tau^{-s}.$$

Note that f is a monotone decreasing function with $f' < 0$ for $\tau > 0$. By the Mean Value Theorem
$$VV''' = \xi^{\frac{1}{\lambda}} f'(W)V^2$$

for some W between $\widetilde{U}_2$ and U_1. Thus
$$VV''' \leq 0 \text{ on } [0, \eta_1). \tag{29}$$

Set
$$h(\xi) = V(\xi)V''(\xi) - \frac{1}{2}V'^2(\xi).$$

Then
$$h(0) = (c-1)V''(0) \text{ and } h' = VV''' \leq 0 \text{ on } [0, \eta_1),$$

i.e., h is a monotone decreasing function on $[0, \eta_1]$. Using the asymptotic expansion (15) we find that for sufficiently large $\xi < \eta_1$
$$h(\xi) \sim K(\eta_1 - \xi)^{\frac{2(1-a)}{2+a}}$$

for some constant K. Since $a < 1$ it follows that

$$h(\eta_1^-) = 0.$$

If $h(0) \le 0$ then h decreasing and $h(\eta_1^-) = 0$ imply that $h \equiv 0$ on $[0, \eta_1]$. Thus

$$h' = \xi^{\frac{1}{\lambda}}(\widetilde{U}_2^{-s} - U_1^{-s})(\widetilde{U}_2 - U_1) = 0 \text{ on } [0, \eta_1)$$

and this implies that $\widetilde{U}_2 = U_1$ on $[0, \eta_1)$. In particular, $c = \widetilde{U}_2(0) = U_1(0) = 1$. It follows that $\eta_1 = \eta_2$ so that

$$U_2(\xi) = \widetilde{U}_2(\xi) = U_1(\xi) \text{ on } [0, \eta_1].$$

Suppose that $h(0) > 0$. Together with $h(\eta_1^-) = 0$ and the fact that h is decreasing this implies that $h \ge 0$ on $[0, \eta_1]$. Therefore

$$VV'' \ge \frac{1}{2}V'^2 \ge 0 \text{ on } [0, \eta_1]. \tag{30}$$

Moreover, it follows from $h(0) > 0$ that $c > 1$ and $V''(0) > 0$. Hence there exists a $\zeta_1 \in (0, \eta_1)$ such that

$$V \ge c - 1 > 0 \text{ on } [0, \zeta_1] \text{ and } V'(\zeta_1) > 0.$$

On the other hand, $V(\eta_1) = 0$ means that V, which is initially increasing, must start to decrease somewhere in (ζ_1, η_1). Thus there exists a $\zeta_2 \in (\zeta_1, \eta_1)$ such that

$$V > 0 \text{ on } [0, \zeta_2] \text{ and } V'(\zeta_2) < 0.$$

It follows that there exists a $\zeta_3 \in (\zeta_1, \zeta_2)$ such that

$$V(\zeta_3)V''(\zeta_3) < 0$$

and this contradicts (30). We conclude that we must have $h(0) \le 0$ and, as we have shown above, this implies that $U_1 \equiv U_2$.

3 The radially symmetric case

In this section we will briefly describe how the proof of existence and uniqueness for the 1-D model extends to radially symmetric solutions. In this case, the evolution equation is (cf. [7]):

$$u_t + \frac{1}{r}\left(ru^{\lambda+2}\left|\left(u_{rr} + \frac{u_r}{r}\right)_r\right|^{\lambda-1}\left(u_{rr} + \frac{u_r}{r}\right)_r\right)_r = 0$$

and the self-similar solutions are of the form

$$u(r, t) = \frac{A}{t^{2\beta}}U\left(\frac{r}{t^\beta}\right)$$

11

with

$$\beta = \frac{1}{7\lambda + 3}$$

and $U(\eta)$ satisfying the following ordinary differential equation:

$$U^{\lambda+2}\left(\left(U'' + \frac{U'}{\eta}\right)'\right)^{\lambda} = \eta U \ . \tag{31}$$

Analogously to (6) we impose

$$U(0) = 1, \qquad U'(0) = 0, \qquad \lim_{\eta \to 0^+} \frac{1}{\eta}(\eta U')' = -\kappa \ . \tag{32}$$

After a suitable rescaling, the problem (31), (32) becomes

$$z^{1+a}\left(z'' + \frac{z'}{x}\right)' = \gamma x^a \ , \tag{33}$$

$$z(0) = 1, z'(0) = 0, \ \lim_{x \to 0^+} \frac{1}{x}(xz')' = -1. \tag{34}$$

We will denote the solutions to this problem by $z(x, \gamma)$. One can easily check that the asymptotics near the contact line are identical to the one dimensional case, and in particular, for the zero contact angle solutions are given by Eq. (15) (cf. [7]). When $\gamma = 0$, there exists an explicit solution to (33), (34) given by

$$z(x, 0) = 1 - \frac{x^2}{4} \ .$$

If $\gamma < 0$ then it is simple to show that z', z'', z''' are all negative whenever $z > 0$, implying the existence on compactly supported solutions to (33), (34) in this case.

If $\gamma > 0$ then one can write

$$\left(\frac{1}{x}(xz')'\right)' = \gamma \frac{x^a}{z^{1+a}} \ .$$

Noticing that

$$\left(\frac{1}{x}(xz')'\right)' = 8\sqrt{y}\frac{d^2}{dy^2}\left(y\frac{dz}{dy}\right) \ ,$$

where $y = x^2$ it is clear that function $y\frac{dz}{dy}$ is concave up. At the origin $y\frac{dz}{dy} = 0$ and $\frac{d}{dy}\left(y\frac{dz}{dy}\right) < 0$. Hence, $\frac{dz}{dy}$ has at most one zero for $y > 0$. This implies $z(x, \gamma)$ can either decrease monotonically to zero or grow monotonically after a minimum.

The use of implicit function theorem is analogous to the 1-D case and shows the existence of zero contact angle solutions as well as non-zero contact angle solutions. Since the proof follows the same lines, we omit the details.

The proof of uniqueness of the zero contact angle solution is similar to the 1-D case except for the fact that the equivalent to inequality (29) is now

$$V\left(\frac{1}{\eta}\left(\eta V'\right)'\right)' = (V''' + \frac{V''}{\eta} - \frac{V'}{\eta^2})V \leq 0 . \tag{35}$$

Without loss of generality we can assume that $V(0) > 0$. One can easily verify that $V''(0) = \frac{1}{2}\lim_{\eta\to 0^+}\frac{(\eta V')'}{\eta}$.

Case 1: $\lim_{\eta\to 0^+}\frac{(\eta V')'}{\eta} < 0$. Using the asymptotic expansion (15) we can conclude that there exists an interval (η_2, η_1) where $V(\eta)$ and $V''(\eta)$ have a given sign and, moreover, $V(\eta)V(\eta)'' > 0$ in that interval.

Let η_1^- be a point of the interval sufficiently close to η_1. Let η^* be the inflection point closest to η_1. Given the regularity of V, since $V''(0) < 0$, $V(0) > 0$ and $V(\eta)V(\eta)'' > 0$ for $\eta \in (\eta_2, \eta_1)$ such a point must exist and $V''(\eta^*) = 0$.

We will show that $V(\eta^*)V(\eta_1^-) > 0$ for $\eta^* < \eta_1^-$. The proof applies to the case in which $V(\eta_1^-) > 0$ (the proof in the case $V(\eta_1^-) < 0$ is analogous). Suppose that $V(\eta^*) < 0$. This implies the existence of some point $\eta^{**} \in (\eta^*, \eta_1)$ at which a local maximum of V is achieved and $V''(\eta^{**}) < 0$. Hence there must exist and inflection point in (η^{**}, η_1) which contradicts the fact that η^* is the inflection point closest to η_1.

Since $V''(\eta^*)$ is negative for $\eta \in (\eta^* - \varepsilon, \eta^*)$ and positive for $\eta \in (\eta^*, \eta^* + \varepsilon)$ we have that VV'' is increasing at η^* and hence $\frac{d}{d\eta}(VV'')\Big|_{\eta^*} = V(\eta^*)V'''(\eta^*) + V'(\eta^*)V''(\eta^*) = V(\eta^*)V'''(\eta^*) \geq 0$ and $V'(\eta^*)V(\eta^*) < 0$. This implies $(V'''(\eta^*) + \frac{V''(\eta^*)}{\eta^*} - \frac{V'(\eta^*)}{\eta^{*2}})V(\eta^*) > 0$ which contradicts (35).

Case 2: $\lim_{\eta\to 0^+}\frac{(\eta V')'}{\eta} > 0$. In the interval $(0, \eta_1)$ there must exists a local maximum η_2 for V and we can repeat the argument of the previous case looking for an inflection point at the left of η_2 instead of η_1.

4   Conclusions

In this paper we have established the existence of solutions representing the spreading of drops in a model for the capillary spreading of Non-Newtonian fluids of power-law rheology. The solutions depend on the rheology exponent λ. Here we prove that when $\lambda > 1$ then for a given mass of fluid, there exists only one solution with zero contact angle at the horizontal substrate and infinitely many solutions with a finite contact angle. Both the spreading rate and the height of the drop are power laws whose respective exponents depend on λ. For the case when $\lambda < 1$ such spreading solutions do not exist. Both results are valid for both planarly and radially symmetric drops.

The results presented here pose a number of interesting questions:

1. Does these solutions represent the intermediate asymptotics of the spreading of a compactly supported drop?. In other words, does the solution as $t \to \infty$ depend on the initial shape of the drop?.

2. Is there a selection criterion between zero and non-zero contact angle solutions?. This is perhaps an stability problem that can be recast in the following way: which of these solutions is stable with respect to small perturbations of the initial data?.

5 Appendix: the traveling wave solutions in 1-D

In this section we study the existence and asymptotic behavior of traveling wave solutions. These solutions are relevant, since they allow the determination of the local behavior of moving fronts near the interface. They are of the form

$$u(x,t) = f(x + ct) .\tag{36}$$

Substituting (36) in (1) we get the equation

$$cf + f^{\lambda+2}|f_{\xi\xi\xi}|^{\lambda-1}f_{\xi\xi\xi} = K\tag{37}$$

where K is a real constant. The condition (2) forces us to choose $K = 0$. Since

$$|f_{\xi\xi\xi}|^{\lambda-1}f_{\xi\xi\xi} = -cf^{-1-\lambda}$$

one has that $f_{\xi\xi\xi} > 0$ if $c < 0$ and $f_{\xi\xi\xi} < 0$ if $c > 0$. Therefore,

$$|f_{\xi\xi\xi}| = |c|^{\frac{1}{\lambda}} f^{-1-\frac{1}{\lambda}} .$$

and we can remove the factor $|c|^{\frac{1}{\lambda}}$ by performing the change of variable $f \to |c|^{\frac{1}{2\lambda+1}} f$. Given the translation and $\xi \to -\xi$ reflection invariance of the equation we can assume without loss of generality that the solutions representing fronts are defined for $\xi \geq 0$ and the front is located at $\xi = 0$. We will also assume $f_{\xi\xi\xi} < 0$ and arrive to

$$f_{\xi\xi\xi} = -f^{-1-\frac{1}{\lambda}} .\tag{38}$$

Let us define

$$x = f$$
$$y = f^{-\alpha}f_\xi$$
$$z = f^\beta f_{\xi\xi}$$

with $\alpha = \frac{\lambda-1}{3\lambda}$ and $\beta = \frac{\lambda+2}{3\lambda}$ together with the change of variable defined by

$$x^{-(\beta+\alpha)}d\xi = d\xi_1\tag{39}$$

so that the solutions of (38) are orbits of the following third order autonomous dynamical system:

$$x' = xy\tag{40}$$
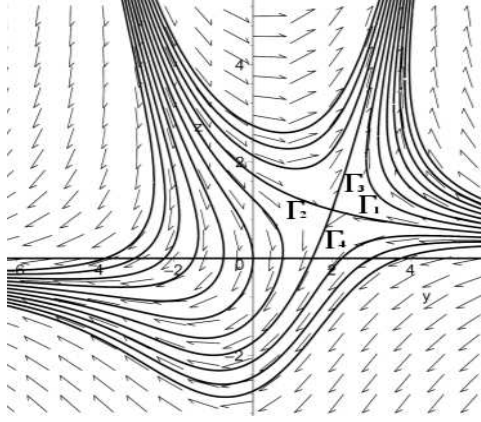$$y' = z - \alpha y^2\tag{41}$$
$$z' = -1 + \beta yz\tag{42}$$

14

Figure 1: The phase plane for (41), (42)

Notice that equations (41), (42) are uncoupled to equation (40) so that the problem reduces to the analysis of the phase plane for equations (41), (42).

There exists a unique equilibrium point at $P : (y, z) = ((\beta\alpha)^{-\frac{1}{3}}, \alpha^{\frac{1}{3}}\beta^{-\frac{2}{3}})$. This point is a saddle point as one can easily check. Hence, there exists an stable manifold Σ_1 through P as well as an unstable manifold Σ_2. Σ_1 consists on the union of two separatrices Γ_1, Γ_2 and Σ_2 on the union of another two separatrices Γ_3, Γ_4 (see figure 1). There exist four different asymptotic behaviors for the trajectories as $|y|$ tends to infinity and each of these asymptotic behaviors corresponds to the one exhibited by Γ_1, Γ_2, Γ_3, Γ_4 respectively. Now we proceed to describe these behaviors in detail:

1) Γ_1 and Γ_4 are such that $y \to \pm\infty$ and $z \to 0^\pm$ respectively. Then, from (41), (42) one obtains

$$y' \sim -\alpha y^2 \ , \ z' \sim -1 + \beta yz$$

from which it follows

$$\frac{dz}{dy} \sim \frac{1 - \beta yz}{\alpha y^2} \Rightarrow z \sim Cy^{-\frac{\beta}{\alpha}} - \frac{1}{(\alpha - \beta)y} \sim -\frac{1}{(\alpha - \beta)y} \text{ as } y \to \pm\infty \quad (43)$$

since $\frac{\beta}{\alpha} = \frac{\lambda+2}{\lambda-2} > 1$.

2) Γ_3 and Γ_2 are such that $z \to +\infty$ and $y \to \pm\infty$ respectively. Then, from (41), (42) one obtains

$$z' \sim \beta yz \ , \ y' \sim z - \alpha y^2$$

from which it follows

$$\frac{dz}{dy} \sim \frac{\beta yz}{z - \alpha y^2} \Rightarrow (\alpha + \frac{\beta}{2})y^2 \sim C_1 z^{-\frac{2\alpha}{\beta}} + z \Rightarrow z \sim (\alpha + \frac{\beta}{2})y^2 \text{ as } y \to \pm\infty \quad (44)$$

since $\frac{2\alpha}{\beta} > 0$.

15

Once we have discussed the asymptotic properties of the trajectories in the phase plane corresponding to (41), (42), we proceed to study the behavior of the trajectories in the three-dimensional phase space for (40), (41), (42). Every manifold of the form $\mathbb{R} \times \Gamma$ with Γ being a trajectory in the phase plane, is invariant. As we will see, the most interesting case corresponds to $\Gamma = \Gamma_i$ $(i = 1, 2, 3, 4)$. The invariant manifolds $\Pi_i \equiv \mathbb{R} \times \Gamma_i$ intersect at the points $\mathbb{R} \times P$ which form the trajectory $(x, y, z) = (e^{(\beta\alpha)^{-\frac{1}{3}}\xi_1}, (\beta\alpha)^{-\frac{1}{3}}, \alpha^{\frac{1}{3}}\beta^{-\frac{2}{3}})$. The behavior of the trajectories on Π_i $(i = 1, 2, 3, 4)$ are rather different. The trajectories in Π_1 and Π_2 approach asymptotically $\mathbb{R} \times P$ as $x \to +\infty$. From equations (40), (41) and the asymptotic behavior computed in formula (43) it follows that $y \simeq Kx^{-\alpha}$ as $x \to 0^+$ (K an arbitrary positive constant) for the trajectories in Π_1. Analogously, from equations (40), (41) and the asymptotic behavior computed in formula (44) it follows that $y \simeq Kx^{\frac{\beta}{2}}$ when $y \to -\infty$ (K an arbitrary negative constant) for the trajectories in Π_2. The trajectories in Π_3 and Π_4 start at $(x, y, z) = (0, (\beta\alpha)^{-\frac{1}{3}}, \alpha^{\frac{1}{3}}\beta^{-\frac{2}{3}})$. Analogously to the trajectories in Π_2 and Π_1, the trajectories in Π_3 are such that $y \simeq Kx^{\frac{\beta}{2}}$ when $x \to \infty$ (K an arbitrary negative constant) and the trajectories in Π_4 are such that $y \simeq Kx^{-\alpha}$ as $x \to 0^+$ (K an arbitrary positive constant).

Finally, we translate the phase space trajectories described above into solutions of (38) and discuss their physical significance. The trajectory in $\mathbb{R} \times P$ is such that, by (39) and (40), we have

$$\frac{dx}{d\xi} = x^{1-(\beta+\alpha)} (\beta\alpha)^{-\frac{1}{3}}$$

which implies, imposing $x(0) = 0$, that

$$f(\xi) = x(\xi) = \left[(\alpha + \beta)(\beta\alpha)^{-\frac{1}{3}}\right]^{\frac{1}{(\alpha+\beta)}} \xi^{\frac{1}{\alpha+\beta}} \equiv C_\lambda \xi^{\frac{3\lambda}{2\lambda+1}} .$$

The trajectories in Π_1 are such that $f(\xi) \sim C_\lambda \xi^{\frac{3\lambda}{2\lambda+1}}$ as $\xi \to \infty$ and, given that $y \simeq Kx^{-\alpha}$ as $x \to 0^+$, by (39) and (40) one has

$$\frac{dx}{d\xi} = K$$

so that $f(\xi) \sim K\xi$ as $\xi \to 0^+$.

The trajectories in Π_2 are such that $f(\xi) \sim C_\lambda \xi^{\frac{3\lambda}{2\lambda+1}}$ as $\xi \to +\infty$ and $f(\xi) \sim K\xi^2$ as $\xi \to -\infty$. Since there is no front in this case, we do not consider this solution to be physically relevant.

The trajectories in Π_3 are such that $f(\xi) \sim C_\lambda \xi^{\frac{3\lambda}{2\lambda+1}}$ as $\xi \to 0^+$ and $f(\xi) \sim K\xi^2$ as $\xi \to \infty$.

The trajectories in Π_4 are such that $f(\xi) \sim C_\lambda \xi^{\frac{3\lambda}{2\lambda+1}}$ as $\xi \to 0^+$ and $f(\xi) \sim K(\xi_0 - \xi)$ as $\xi \to \xi_0^-$ for some positive ξ_0. They are compactly supported.

In addition there exist another four families of solutions such that the trajectories in phase space approach asymptotically to two of the manifolds Π_i

$(i = 1, 2, 3, 4)$. The first one approaches Π_1 and Π_4 so that behaves linearly at the origin and linearly close to some $\xi_0 > 0$. It is compactly supported and lacks a clear physical significance. The second one approaches Π_4 and Π_2 so that it is linear at the origin and grows quadratically at infinity. For them $y < 0$ which implies that $f'(\xi) < 0$. They represent dewetting solutions. The third one approaches Π_2 and Π_3 and grows quadratically at $\pm\infty$ presenting no fronts. The last one approaches Π_3 and Π_1. It behaves linearly at the origin and grows quadratically at infinity.

To summarize, one has infinitely many solutions that behave linearly at the origin and such that $f(\xi) \sim C_\lambda \xi^{\frac{3\lambda}{2\lambda+1}}$ as $\xi \to \infty$. There is only one solution (which is explicit) such that $f(\xi) \sim C_\lambda \xi^{\frac{3\lambda}{2\lambda+1}}$ at the origin (zero contact angle) and at infinity. There exist infinitely many solutions with zero contact angle at the origin and growing quadratically at infinity and, finally there exist dewetting solutions which are linear at the origin and grow quadratically at infinity. Hence, the only local behaviors near the interface for moving fronts are those with finite contact angle and the one with zero contact angle.

References

[1] Ansini L. and Giacomelli L. "Shear-thinning liquid films: Macroscopic and asymptotic behavior by quasi self-similar solutions", Nonlinearity **15** (6), pp. 2147-2164 (2002).

[2] Barenblatt G. I. "Scaling, Self-Similarity and Intermediate Asymptotics", Cambridge University Press, (1996).

[3] Bernis F. and Ferreira R., "Source-Type Solutions to Thin-Film Equations: The Critical Case", Applied Mathematics Letters **12** pp. 45-50 (1999).

[4] Bernis F., Hulshof J., King J. R., "Dipoles and similarity solutions of the thin film equation in the half-line", Nonlinearity, **13**, no.2, pp. 413-439 (2000).

[5] Bertozzi A. L. and Pugh M., "The lubrication approximation for thin viscous films: regularity and long time behaviour of weak solutions", Comm. Pure. Appl. Math., **49**(2), pp. 85-123 (1996).

[6] Betelu S. I. and Fontelos M. A., "Capillarity driven spreading of power law fluids", Applied Mathematics Letters, in press. (2002).

[7] Betelu S. I. and Fontelos M. A., "Capillarity driven spreading of circular drops of shear-thinning fluid", submitted to Mathematical and Computer Modelling.

[8] R. B. Bird, R. C. Armstrong and O. Hassager *Dynamics of polymeric liquids*, John Wiley and Sons, 1977.

[9] Carre A. and Eustache F., "Spreading Kinetics of Shear-Thinning Fluids in Wetting and Dewetting Modes", Langmuir **16** pp. 2936-2941 (2000).

[10] King J. R., "Two Generalizations of the Thin Film Equation", Mathematical and Computer Modelling, **34** pp. 737-756 (2001).